A short introduction to rough paths

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Rough paths

What? A theory to define integrals/diff eq. driven by irregular paths

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s$$

such that "$x \mapsto y$" is continuous.

Why? To

- define pathwise SDE
- extend SDE to other noises than Brownian ones (fractional, ...)
- develop numerical schemes
- perform machine learning task, ...

Who, When and Where? T. Lyons at Oxford U. in 1998, now world-wide (France, Germany, Japan, ...).

— I —

The intuition behind the rough paths theory: the role of the Lévy area

Stratonovich integrals

Itô/Stratonovich integrals cannot be defined pathwise.

$$\int_0^T f(B_s) \circ \mathrm{d}B_s$$

$$= \mathbb{P}\text{-}\lim_{[u,v]\in\pi} \frac{f(B_u)+f(B_v)}{2}(B_v - B_u) \quad \textbf{(Riemann type)}$$

$$= \mathbb{P}\text{-}\lim \int_0^T f(B_s^\pi)\,\mathrm{d}B_s^\pi \quad \textbf{(Wong-Zakai)}$$

where B^π piecewise linear approximation of B along dyadic partitions π.

Stratonovich integrals

B d-dimensional BM with increments $B_{s,t} := B_t - B_s$

Define

$$K^{i,j}_{[s,t]} = \int_s^t (B^i_r - B^i_s) \circ \mathrm{d}B^j_r = A^{i,j}_{[s,t]} + S^{i,j}_{[s,t]}.$$

Define also

$$\begin{cases} A^{i,j}_{[s,t]} := \frac{1}{2}\left(K^{i,j}_{[s,t]} - K^{j,i}_{[s,t]}\right) & \text{Lévy area} \\ S^{i,j}_{[s,t]} := \frac{1}{2} B^i_{s,t} \times B^j_{s,t} & \text{symmetric part} \end{cases}$$

The symmetric part $S_{[s,t]}$ is a function of the increment $B_{s,t}$, unlike $A_{[s,t]}$.

Stratonovich integrals

Recall that

$$\int_0^T f(B_s) \circ \mathrm{d}B_s = \mathbb{P}\text{-}\lim_{[u,v]\in\pi} \frac{f(B_u)+f(B_v)}{2}(B_v - B_u)$$

$$= \mathbb{P}\text{-}\lim_{[u,v]\in\pi} +f(B_u)(B_v - B_u) + \frac{f(B_v)-f(B_u)}{2}(B_v - B_u)$$

With the Newton formula (think Stratonovich)

$$f_j(B_v) - f_j(B_u) = \sum_{i=1}^{d} \int_u^v \partial_i f_j(B_r) \circ \mathrm{d}B_r^i$$

$$= \sum_{i=1}^{d} \partial_i f_j(B_u) B_{u,v}^i + \text{rem.}$$

$$\Longrightarrow f_j(B)_{u,v} B_{u,v}^j = \sum_{i=1}^{d} \partial_i f_j(B_u) S_{[u,v]}^{i,j} + \text{rem.}$$

Stratonovich integrals

$$I_\pi = \sum_{[u,v]\in\pi} \left(f(B_u)B_{u,v} + \partial_j f_i(B_u) K^{i,j}{}_{[u,v]}\right)$$

$$= \underbrace{\sum_{[u,v]\in\pi} f(B_u)(B_v - B_u) + \frac{1}{2}(f(B_v) - f(B_u))B_{u,v}}_{\text{Stratonovich approx.}} + \text{rem.}$$

$$+ \underbrace{\sum_{[u,v]\in\pi} \partial_j f_i(B_u) A^{i,j}_{[u,v]}}_{\to 0 \text{ in prob. due to a martingale arg}}$$

The theory of rough paths shows that I_π converges almost surely.

The theory also endows the primary role of the Lévy area.

The role played by the Lévy area

B^π piecewise linear interpolation of the Brownian motion

$$I_\pi = \sum_{[u,v]\in\pi} \Big(f(B_u) B_{u,v} + \partial_j f_i(B_u) K^{i,j}_{[u,v]} \Big)$$
$$= \int_0^T f(B^\pi_s)\,\mathrm{d}B^\pi_s + \textbf{rem} \xrightarrow[|\pi|\to 0]{} \int_0^T f(B_s)\circ \mathrm{d}B_s.$$

Choose $d=2$ and $\phi \in \mathcal{C}^1([0,T],\mathbb{R})$.

Use another family of approximation of B along π s.t.

$$\mathbb{E}[A^{1,2}(B^\pi)_{[u,v]} \mid \mathcal{F}^\pi_u] = -\mathbb{E}[A^{2,1}(B^\pi)_{[u,v]} \mid \mathcal{F}^\pi_u] \approx \phi_{u,v}.$$

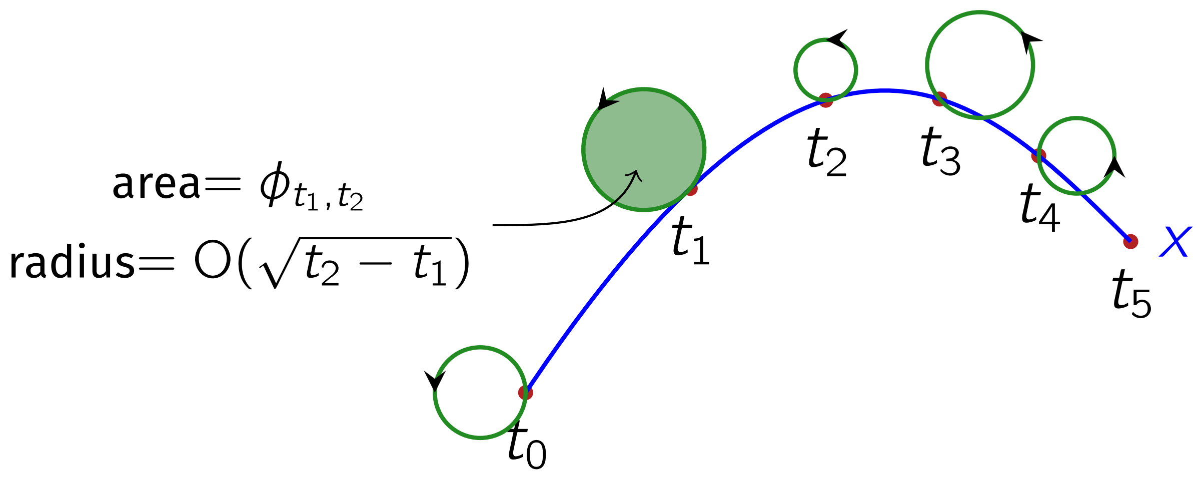
Then (McShane 1972)

$$I_\pi \xrightarrow[|\pi|\to 0]{} \int_0^T f(B_s)\circ \mathrm{d}B_s + \frac{1}{2}\int_0^T (\partial_1 f_2 - \partial_2 f_1)(B_s)\,\mathrm{d}\phi_s.$$

Smooth paths converging to smooth paths

This is not related to the regularity of the path itself, but to the regularity of the approximating sequence.

- $x \in \mathcal{C}^1([0, T], \mathbb{R}^2)$
- extra (=arbitrary) $\phi \in \mathcal{C}^1([0, T], \mathbb{R})$
- $x^\pi = x +$ loops of area $\phi_{u,v}$
- $f = (f_1, f_2)$ smooth differential form



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$$\int_{x^\pi} f = \int_x f + \sum_{[u,v]\in\pi} \int_{\text{loop}} f \overset{\text{Gauss-Green}}{=} \int_x f + \sum_{[u,v]\in\pi} \iint_{\substack{\text{area of}\\ \text{the loop}}} \mathrm{d}f$$

$$\approx \int_x f + \sum_{[u,v]\in\pi} (\partial_1 f_2 - \partial_2 f_1)(x_u)\phi_{v,u}$$

$$\xrightarrow[|\pi|\to 0]{} \int_x f + \int (\partial_1 f_2 - \partial_2 f_1)(x_s)\,\mathrm{d}\phi_s =: \int_{(x,\phi)} f.$$

It's not a bug, its a feature!

- x^π converges uniformly to x.

$\Rightarrow$ The map $x \mapsto \int_x f$ is not continuous w.r.t $\|\cdot\|_\infty$.

- Each x^π converges to x wrt $\|\cdot\|_\alpha$ with $\alpha < 1/2$ (Hölder norm).
- The arbitrary function ϕ corresponds to some exogenous information (called the Lévy area, not necessarily the natural one).

$\Rightarrow$ As $\int_{(x,\phi)} f = \lim \int_{x^\pi} f$, we identify (x, ϕ) with $\{x^\pi\}_\pi$.

- Similarly, when x is irregular, $x \in \mathcal{C}^\alpha$, $1/3 < \alpha \leqslant 1/2$, we identify it with $\{x^\pi\}_\pi$, $x^\pi \in \mathcal{C}^1$ with a prescribed area on $[u, v] \in \pi$.
- The Stratonovich integral corresponds to the natural choice of the Lévy area which statistically disappear in the limit (the Itô integral too, using Itô integration).
- Different choices of $\phi / \{x^\pi\}_\pi$ lead to different integrals.

The intuition behind the theory of rough paths

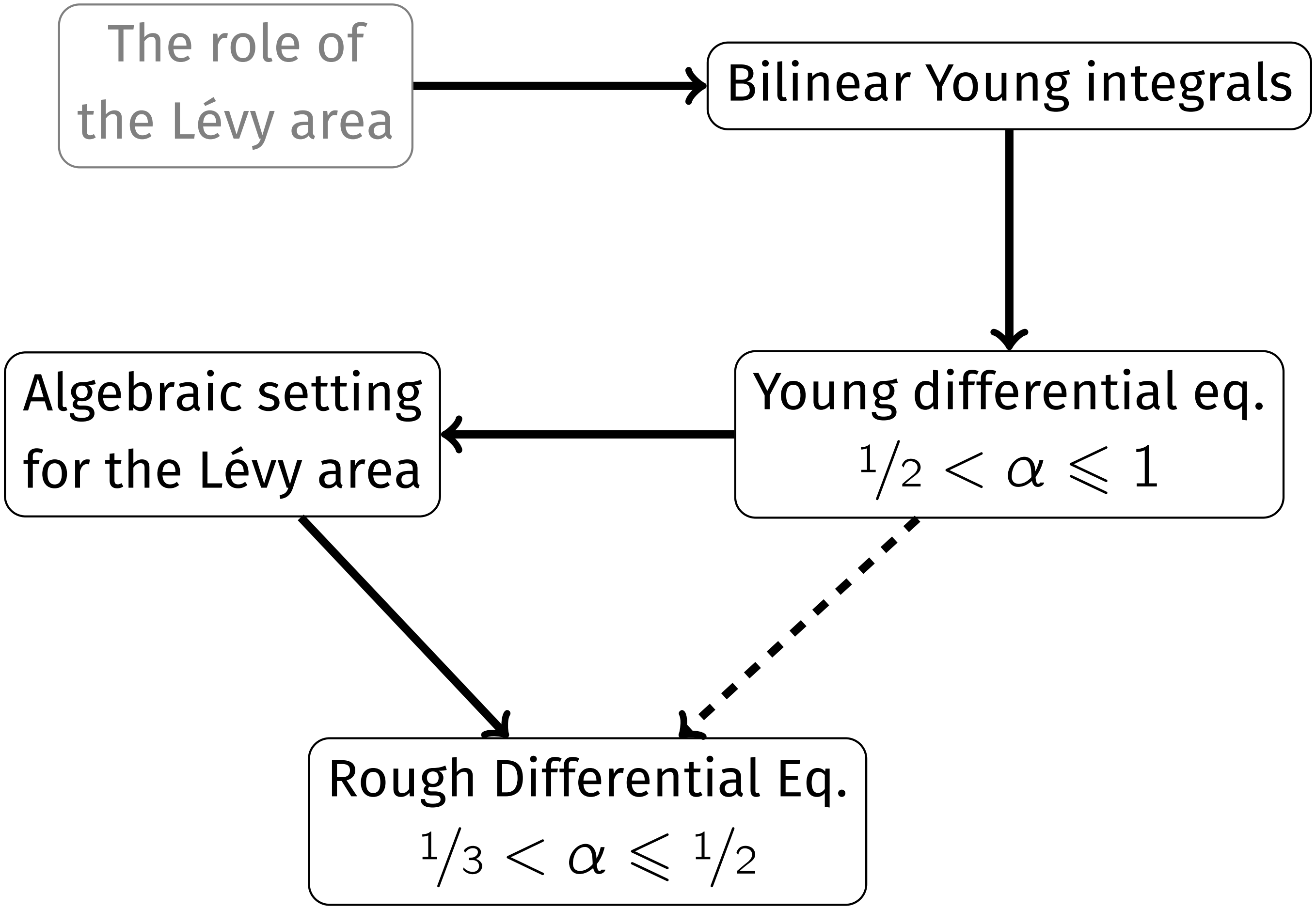
The theory of rough paths lies on the fact that the Lévy area is THE information to define a continuous, natural extension of $x \mapsto \int f(x_s)\,\mathrm{d}x_s$ for $x \in \mathcal{C}^\alpha$, $1/3 < \alpha \leqslant 1/2$.

A **rough path** $\mathbf{x}$ is an enhanced path lying above x that includes such a data.

Once integrals are defined, we can define a consistent theory of differential equations as well.

It could be applied to many stochastic processes. The theory itself is deterministic.

The roadmap



The role of the Lévy area
Bilinear Young integrals
Algebraic setting for the Lévy area
Young differential eq. $1/2 < \alpha \leqslant 1$
Rough Differential Eq. $1/3 < \alpha \leqslant 1/2$

— II —

The technical core for integrals: Young integrals & the additive sewing lemma

Young integrals

Young, 1936/ $x \in \mathcal{C}^\alpha([0,T],\mathbb{R})$, $y \in \mathcal{C}^\beta([0,T],\mathbb{R})$, $\theta := \alpha+\beta > 1$. The Young integral exists as a family bilinear maps

$$(y,x) \mapsto \mathcal{Y}_{s,t}(y,x) := \int_s^t y_r \,\mathrm{d}x_r$$

with

- Chasles' property $\mathcal{Y}_{r,s}(y,x) + \mathcal{Y}_{s,t}(y,x) = \mathcal{Y}_{r,t}(y,x)$
- Regularity $t \mapsto \mathcal{Y}_{0,t}(y,x) \in \mathcal{C}^\alpha([0,T],\mathbb{R})$
- Main control
$$|\mathcal{Y}_{s,t}(y,x) - y_s \cdot x_{s,t}| \leqslant K\|y\|_\beta\|x\|_\alpha(t-s)^\theta. \qquad (\spadesuit)$$

Rem. x, y could take their values in any Banach spaces U and V for which a continuous product $\mathrm{V}\times\mathrm{U} \mapsto \mathrm{W}$ exists.

The additive sewing lemma

Let us set $Y^0_{s,t} = y_s \cdot x_{s,t}$. Since $x_{s,r} + x_{s,t} = x_{s,t}$,

$$Y^0_{r,s} + Y^0_{s,t} - Y^0_{s,r} = y_{r,s} \cdot x_{s,t}.$$

With $s = (t + r)/2$, the midpoint,

$$Y^{n+1}_{r,t} := Y^n_{r,s} + Y^n_{s,t}.$$

It holds that

$$Y^{n+1}_{r,t} - Y^n_{r,t} = Y^n_{r,s} + Y^n_{s,t} - Y^{n-1}_{r,s} - Y^{n-1}_{s,t}.$$

Set

$$V^{n+1}(\delta) := \sup_{|t-s|\leqslant\delta} |Y^{n+1}_{s,t} - Y^n_{s,t}|$$

to obtain

$$V^{n+1}(\delta) \leqslant 2V^n(\delta/2) \leqslant 2^n V^1(\delta/2^n).$$

The additive sewing lemma

Since

$$V^1(\delta) \leqslant \sup_{|t-r|\leqslant\delta} |y_{r,s}| \cdot |x_{s,t}| \leqslant \|y\|_\beta \cdot \|x\|_\alpha \delta^{\alpha+\beta}$$

the sequence $(Y^n_{s,t})_n$ is Cauchy since ($\theta := \alpha + \beta > 1$)

$$V^{n+1}(\delta) \leqslant \frac{1}{2^{n(\theta-1)}} \|y\|_\beta \cdot \|x\|_\alpha \text{ and } \sum_{n\geqslant 0} \frac{1}{2^{n(\theta-1)}} < +\infty.$$

With a bit more analysis,

- $Y^n_{s,t}$ converges to $Y^\infty_{s,t}$ for each (s,t) with

$$Y^\infty_{r,s} + Y^\infty_{s,t} = Y^\infty_{r,t}, \quad \forall (r,s,t). \tag{♣}$$

- For each (s,t),

$$|Y^\infty_{s,t} - Y^0_{s,t}| \leqslant K \|y\|_\beta \cdot \|x\|_\alpha (t-s)^\theta \tag{♠}$$

- Y^∞ is the only family satisfying (♠)-(♣).

The additive sewing lemma

- Y^∞ is the Young integral. It satisfies

$$Y^\infty_{s,t} = \lim_{|\pi|\to 0} \sum_{[u,v]\subset\pi\cap[s,t]} y_u \cdot x_{u,v}.$$

- There are several proofs for this results. They all follow the same general scheme. Here: D. Feyel, A. de la Pradelle & G. Mokobodzki 2008.
- The exact form of Y^0 does not matter. What matters is that

$$|Y^0_{r,s,t}| \leqslant C\varpi(\omega_{r,t}) \text{ with } Y^0_{r,s,t} := Y^0_{r,s} + Y^0_{s,t} - Y^0_{s,t}$$

$$\text{and } \varpi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ increasing }, 2\varpi\left(\frac{\delta}{2}\right) \leqslant \kappa\varpi(\delta), \kappa < 1.$$

- The additive sewing lemma maps continuously $Y^0 \mapsto Y^\infty$ with

$$|Y^\infty_{s,t} - Y^0_{s,t}| \leqslant LC\varpi(\omega_{s,t}) \text{ and } Y^\infty_{r,s,t} = 0.$$

The discrete additive sewing lemma (alternative way)

A.M. Davie 2010/ Let $\{\phi_{i,j}\}_{i=0,\dots,n}$ be such that

$$|\phi_{i,j} + \phi_{j,k} - \phi_{i,k}| \leqslant A\varpi(k-i), \ \forall i \leqslant j \leqslant k.$$

Then there exists $L > 0$ (universal) large enough such that

$$|\psi_{i,k} - \phi_{i,k}| \leqslant L\varpi(k-i) \text{ with } \psi_{i,k} = \sum_{j=i}^{k-1} \phi_{j,j+1}.$$

Proof: Use a doubling argument in a recurrence over $k-i$ with L satisfying $\kappa L + A \leqslant L$ and a "midpoint" j:

$$\psi_{i,k} - \phi_{i,k} = \underbrace{\underbrace{\psi_{i,j} - \phi_{i,j}}_{\leqslant L\varpi(k-j)} + \underbrace{\psi_{i,j} - \phi_{i,k}}_{\leqslant L\varpi(j-i)}}_{\leqslant \kappa L\varpi(k-i)} + \underbrace{\phi_{i,j,k}}_{\leqslant A\varpi(k-i)}.$$

— III —

Toward solving differential equations: Young differential equations

The Omega lemma

Omega lemma, L. Coutin & AL 2018/

- Let $f \in \mathcal{C}^{k+\gamma}(\mathbb{R}^n, \mathbb{R}^m)$, $0 < \gamma \leqslant 1$ (f is k differentiable with bounded der. and $\nabla^k f \in \mathcal{C}^\gamma$).
- Set $\Omega f(x) := (f(x_t))_{t\in[0,T]}$ for $x \in \mathcal{C}^\alpha([0,T], \mathbb{R}^n)$.

$\Rightarrow$ If $k = 0$ then $\Omega f : \mathcal{C}^\alpha \to \mathcal{C}^{\alpha\gamma}$.

$\Rightarrow$ If $k > 0$, then Ωf and is locally $\mathcal{C}^{k+\gamma(1-\kappa)}$-Hölder continuous from $\mathcal{C}^\alpha$ to $\mathcal{C}^{\alpha\kappa\gamma}$ for $0 < \kappa < 1$.

Corollary. If $f \in \mathcal{C}^\gamma$, then

$$\mathcal{F} : x \in \mathcal{C}^\alpha \mapsto \left(\int_0^t f(x_s)\,\mathrm{d}x_s\right)_{t\in[0,T]} \in \mathcal{C}^\alpha$$

is well defined and continuous provided that $\alpha(1+\gamma) > 1$, hence $\alpha > 1/2$ at best.

Stability of balls

We work with the following balls (relatively compact thanks to Ascoli-Arzelà):

$$B(r,R) := \left\{ y \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d) \;\middle|\; |y_0| \leqslant r,\ \|y\|_\alpha \leqslant R \right\}$$

We will adjust r, R and T.

When $y \in B(r,R)$,

$$\|y\|_\infty \leqslant |y_0| + \|y\|_\alpha \underbrace{T^\alpha}_{\text{offers some flexibility}}$$

Stability of balls

For $f \in \mathcal{C}^{k+\gamma}$, $k \geqslant 0$, bounded (to simplify) with bounded derivatives

$$\beta = \begin{cases} \alpha\gamma & \text{if } k = 0 \\ \alpha & \text{if } k \geqslant 1 \end{cases} \text{ and } \gamma' = \begin{cases} \gamma & \text{if } k = 0, \\ 1 & \text{if } k \geqslant 1. \end{cases}$$

Recall that Ωf

- maps $\mathcal{C}^{\alpha}$ to $\mathcal{C}^{\beta}$
- is $\mathcal{C}^{k+(1-\kappa)\gamma}$ from $\mathcal{C}^{\alpha}$ to $\mathcal{C}^{\kappa\beta}$.

Then

$$\Omega f(B(r, R)) \subset B(r', R')$$

with

$$r' = \|f\|_{\infty} + \|f\|_{\gamma'} r^{\gamma'} \text{ and } R' = \|f\|_{\gamma'} R^{\gamma'}.$$

Young differential equations

A Young differential equation is

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s = a + \mathcal{Y}(\Omega f(y), x)_t,\ {\scriptstyle t\in[0,T]} \qquad \text{(YDE)}$$

with the bilinear map

$$\mathcal{Y}(z, x)_t = \int_0^t z_s\,\mathrm{d}x_s \text{ with } z \in \mathcal{C}^\beta, x \in \mathcal{C}^\alpha.$$

We then “seperate” the study of Ωf and $\mathcal{Y}$:

- Ωf is been studied through the Omega lemma.
- $\mathcal{Y}$ is bilinear (hence $\mathcal{C}^\infty$).
- We study (YDE) as functional, abstract equation.

Young differential equations

Fundamental inequality on Young integral

$$|\mathcal{Y}(z,x)_{s,t} - z_s \cdot x_{s,t}| \leqslant C\|z\|_\beta \times \|x\|_\alpha |t-s|^{\alpha+\beta}$$

$$\implies \|\mathcal{Y}(z,x)\|_\alpha \leqslant \|z\|_\infty \cdot \|x\|_\alpha + C\|z\|_\beta \cdot \|x\|_\alpha T^\beta$$

$$\leqslant |z_0| \cdot \|x\|_\alpha + (1+C)\|z\|_\beta \cdot \|x\|_\alpha T^\beta$$

If $z \in B(r_z, R_z)$, $x \in B(r_x, R_x)$, $|a| \leqslant r_a$ then

$$\|a + \mathcal{Y}(z,x)\|_\alpha \leqslant r_z R_x + (1+C) R_z R_x T^\beta.$$

If $y \in B(r_y, R_y)$, then $\Omega f(y) \in B(r_z, R_z)$ function of $\|f\|_\infty$ and $\|f\|_{\gamma'}$.

Thus,

$$R' = \|a + \mathcal{Y}(\Omega f(y), z)\|_\alpha \leqslant r_z R_x + (1+C)\|f\|_{\gamma'} R_y^{\gamma'} R_x T^\beta.$$

We tune T to get $R' \leqslant R_y$, and choose $r_y \geqslant |a|$.

Young differential equations: existence

So far, we have proved that

$$y \in B(r_y, R_y) \implies a + \mathcal{Y}(\Omega f(y), x) \in B(r_y, R_y)$$

for a proper choice of T (short time) which depends on r_x, R_x, $\|f\|_{\gamma'}$ and $\|f\|_\infty$. We can easily pass from short time to any time.

Existence/

Let $f \in \mathcal{C}^\gamma$, $x \in \mathcal{C}^\alpha([0,T], \mathbb{R}^d)$, $\alpha(1+\gamma) > 1$. Then there exists a solution to

$$y = a + \mathcal{Y}(\Omega f(y), x)$$

thanks to a Schauder fixed point theorem.
This solution is not necessarily unique (A.M. Davie 2010).

Young differential equations: uniqueness

When Ωf is Lipschitz continuous from $\mathcal{C}^\alpha$ to $\mathcal{C}^\beta$, when $y_0 = z_0$,

$$\begin{aligned}\|\mathcal{Y}(\Omega f(y), x) - \mathcal{Y}(\Omega f(z), x)\|_\alpha &= \|\mathcal{Y}(\Omega f(y) - \Omega f(z), x)\|_\alpha \\ &\leqslant C\|f\|_{\mathrm{Lip}}\|y\|_\alpha \cdot \|x\|_\alpha T^\beta.\end{aligned}$$

Again, we may tune T to get a contractive operator $y \mapsto \mathcal{Y}(\Omega f(y), x)$.

Uniqueness/

When $f \in \mathcal{C}^{1+\gamma}$, $x \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$, $\alpha(1+\gamma) > 1$, there exists a unique solution to

$$y = a + \mathcal{Y}(\Omega f(y), x)$$

thanks to a Banach fixed point theorem.

This works even in an infinite dimensional space.

Young differential equations: regularity

Assume that $\Omega f \in \mathcal{C}^{1+\lambda}$. The solution of the YDE is solution to the fixed point

$$y = a + \mathcal{Y}(\Omega f(y), x).$$

This solution is unique.

Def. The map $\mathcal{I} : x \mapsto y$ is called the Itô map.

We use the Implicit Function Theorem to relate the regularity of Ωf to the one of $(a, f, x) \mapsto y$.

The Implicit Functions Theorem

Let $\mathcal{F}(y, \lambda)$ be locally of class $\mathcal{C}^\mu$ wrt (y, λ), λ is a parameter. Assume that for (y_0, λ_0),

$$\mathcal{F}(y_0, \lambda_0) = 0$$

and

$$\mathrm{D}_y\mathcal{F}(y_0, \lambda_0) \text{ is invertible}.$$

Then $\mathcal{F}$ is locally a $\mathcal{C}^\mu$-diffeomorphism around from a neighborhood V of (y_0, λ_0) to a neighborhood W of 0 with inverse $\mathcal{G}$ satisfying

$$\mathcal{F}(\mathcal{G}(w, \lambda), \lambda) = w, \ w \in W.$$

This theorem works in Banach spaces.

The Implicit Functions Theorem: Application to YDE

We consider

$$y = a + \mathcal{Y}(\Omega f(y), x) + b \text{ with } x, b \in \mathcal{C}^\alpha,\ \ f \in \mathcal{C}^{1+\gamma}.$$

The parameters are $\lambda = (a, f, x, b)$.
The map $\mathcal{F} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ is then

$$\mathcal{F}(y, \lambda) = y - a - \mathcal{Y}(\Omega f(y), x) - b.$$

Showing that

$$\mathrm{D}_y \mathcal{F}(y, \lambda) \cdot z = z - \mathcal{Y}(\Omega \mathrm{D}_y f(y) \cdot z, x)$$

is invertible around a given (a, f, x, b) is equivalent in solving the linear YDE

$$z - \mathcal{Y}(\Omega \mathrm{D}_y f(y) \cdot z, x) = d$$

for any $d \in \mathcal{C}^\alpha$. This is always possible.

Young differential equations: regularity

Main idea: The regularity of $(a, x, f, b) \mapsto y$ depends on the regularity of Ωf.

Regularity, L. Coutin & AL 2018/

Let $f \in \mathcal{C}^{1+\gamma}$, $x, b \in \mathcal{C}^{\alpha}([0, T], \mathbb{R}^d)$, $\alpha(1+\gamma) > 1$.
The map $(a, f, x, b) \mapsto y$ with y solution to

$$y = a + \mathcal{Y}(\Omega f(y), x) + b$$

is locally of class $\mathcal{C}^{1+(1-\kappa)\gamma}$ wrt all its parameters.

To summarize (Young case $1/2 < \alpha \leqslant 1$)

	ODE	YDE
"Cauchy-Peano" existence non uniqueness *a priori*	f continuous	$f \in \mathcal{C}^\gamma$ $\alpha(1+\gamma) > 1$ sol in $\mathcal{C}^\alpha$
"Cauchy-Lipschitz" uniqueness Lipschitz flow conv. of Euler scheme	f Lipschitz $f \in \mathcal{C}^k \Rightarrow \mathcal{I} \in \mathcal{C}^k$	$f \in \mathcal{C}^{1+\gamma}$ $\alpha(1+\gamma) > 1$ $\Rightarrow \mathcal{I} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ $f \in \mathcal{C}^{k+\gamma} \Rightarrow \mathcal{I} \in \mathcal{C}^{k+\epsilon\gamma}$

- This classification is essentially sharp.
- Several solution may exists unless $f \in \mathcal{C}^{1+\gamma}$ (Davie 2010).
- Global solution (any time horizon) exists.
- A measurable flow may exists even when the solution is not unique (A. Brault & AL 2018).

— IV —

Young differential equations: convergence of numerical approximations

Discrete YDE

The above approaches work *verbatim* when $\mathcal{Y}$ is replaced by the discrete integral

$$\mathcal{Y}_\pi(y,x) = \sum_\pi y_u \cdot x_{u,v} \text{ for } (y,x) \in \mathcal{C}^\beta_\pi \times \mathcal{C}^\alpha_\pi$$

and $\mathcal{C}^\alpha$ is replaced by

$$\mathcal{C}^\alpha_\pi = \{y : \pi \to \mathbb{R}^m\}$$

with

$$\|y\|_\alpha = \sup_{\substack{s,t\in\pi \\ s<t}} \frac{|y_t - y_s|}{(t-s)^\alpha}$$

and

$$\Omega f(y) = \{f(y_s) \mid s \in \pi\}.$$

Why considering discrete YDE?

When y^π is a fixed point,

$$y^\pi = a + \mathcal{Y}_\pi(\Omega f(y^\pi), x)$$

then for $[u, v] \in \pi$,

$$y^\pi_v - y^\pi_u = f(y^\pi_u) \cdot x_{u,v}.$$

Thus, when $\pi = \{t_0 < t_1 < \cdots < t_m\}$, y^π is constructed iteratively by

$$y^\pi_{i+1} = y^\pi_i + f(y^\pi_i) \cdot x_{i,i+1}, \ y^\pi_0 = a$$

with the convention $z_i := z_{t_i}$.

The fixed point is explicitly constructed through a numerical, explicit scheme (Euler scheme). Using controls uniform in π, the Ascoli-Arzelà theorem replaces the Schauder/Banach fixed point theorem.

Why considering discrete YDE?

Euler scheme / A.M. Davie 2010

For $f \in \mathcal{C}^\gamma$ and $x \in \mathcal{C}^\alpha$ with $\alpha(1+\gamma) > 1$, there exists a constant L depending only on $\|x\|_\alpha$, T and $\|f\|_\gamma$ such that

$$|y_t^\pi - y_s^\pi - f(y_s^\pi) \cdot x_{s,t}| \leqslant L|t-s|^{\alpha(1+\gamma)}, \ \forall s \leqslant t, (s,t) \in \pi$$

for any partition π.

Therefore, any limit of $\{y^\pi\}$ satisfies

$$|y_t - y_s - f(y_s) \cdot x_{s,t}| \leqslant L|t-s|^{\alpha(1+\gamma)}, \ \forall s \leqslant t \qquad (\clubsuit)$$

Consequences of the construction of A.M. Davie

1. (♣) may serve as an alternative definition of solution.
2. When $f \in \mathcal{C}^{1+\gamma}$, the Euler scheme is locally $\mathcal{C}^{1+\lambda}$, $\lambda < \gamma$ wrt the parameters.
3. When $f \in \mathcal{C}^{1+\gamma}$, existence and uniqueness may be proved in the infinite dimensional setting.
4. When $f \in \mathcal{C}^{1+\gamma}$, the rate of convergence of y^π toward y may be quantified.
5. The construction encompasses the importance of the approximation of the flow (developed later)
$$\phi_{t,s}(a) = a + f(a) \cdot x_{s,t}.$$
6. When $f \in \mathcal{C}^{1+\gamma}$, a flow of $\mathcal{C}^{1+\lambda}_{\mathrm{loc}}$-diffeomorphism exists, that is a family $\psi_{t,s}$ with
$$\psi_{t,s}(\psi_{s,r}(a)) = \psi_{t,r}(a) \text{ and } |\psi_{t,s}(a) - \phi_{t,s}(a)| \leqslant C|t-s|^{\alpha(1+\gamma)}.$$

— V —

Multiplicative functionals:
From Chen series to rough paths

Linear equations and multiplicative functionals

- $(\mathfrak{B}, +, \cdot)$ Banach algebra ($\equiv$ vector space with a product), 1 unit element, $\|ab\| \leqslant \|a\| \times \|b\|$

  Example: Space of linear, bounded operators
- $\alpha > 1/2$, $a \in \mathcal{C}^\alpha([0, T], \mathfrak{B})$.
- We consider the family of linear equations

$$y_{r,t} = 1 + \int_r^t y_{r,s}\,\mathrm{d}a_s, \quad \forall r \leqslant t$$

$\Rightarrow$ Each $y_{r,\cdot} \in \mathcal{C}^\alpha([0, T], \mathfrak{B})$ is unique (Young differential equation).

$\Rightarrow$ The multiplicative property is satisfied

$$y_{r,s} \cdot y_{s,t} = y_{r,t}, \quad \forall r \leqslant s \leqslant t \tag{MF}$$

$\{y_{s,t}\}_{t,s}$ a is resolvent/propagator/semi-group

Multiplicative Young integrals

Let us set

$$z_{t,s} = \prod_s^t (1 + \, \mathrm{d}a_r) := \lim_{|\pi|\to 0} \prod_{[u,v]\in\pi\cap[s,t]} (1 + a_{u,v}).$$

Remark that

$$|(1 + a_{r,s})(1 + a_{s,t}) - (1 + a_{r,t})| = |a_{r,s}a_{s,t}| \leqslant \|a\|_\alpha (t - r)^{2\alpha}.$$

Thanks to the Multiplicative Sewing Lemma (a variant of the Additive Sewing Lemma), z exists, is unique and satisfies

$$z_{r,s} \cdot z_{s,t} = z_{r,t}, \ \forall r\leqslant s\leqslant t$$

$$|z_{s,t} - 1 - a_{s,t}| \leqslant C(t - s)^{2\alpha}.$$

Multiplicative Young integrals

We have $y = z$ since

$$0 = y_{r,t} - 1 - \int_r^t y_{s,r}\,\mathrm{d}a_s$$

$$= y_{s,r}\left(y_{s,t} - 1 - a_{s,t}\right) - \underbrace{\int_s^t (y_{r,u} - y_{r,s})\,\mathrm{d}a_u}_{|\cdot|\leqslant C(t-s)^{2\alpha}}.$$

The solution of the linear equation $y_{r,t} = 1 + \int_r^t y_{r,s}\,\mathrm{d}a_s$ is given by the multiplicative integral $y_{r,t} = \prod_r^t (1 + \,\mathrm{d}a_s)$.

Besides, $y_{r,s}$ is invertible in $\mathfrak{B}$ with inverse

$$z_{t,r} = 1 - \int_r^t \mathrm{d}a_s \cdot z_{s,r} \text{ and } z_{t,s} \cdot z_{s,r} = z_{t,r}.$$

A series à la Dyson

As $y_{s,s} = 1$ and

$$y_{r,t} = 1 + \int_r^t y_{r,u}\,\mathrm{d}a_u$$

$$= y_{r,t} + a_{r,t} + \int_r^t (y_{r,s} - 1)\,\mathrm{d}a_s$$

$$= 1 + a_{r,t} + \int_r^t \int_r^{t_1} \mathrm{d}a_{t_2}\,\mathrm{d}a_{t_1} + \int_r^t \int_r^{t_1} (y_{s,t_2} - 1)\,\mathrm{d}a_{t_2}\,\mathrm{d}a_{t_1}$$

$$= \cdots$$

$=$ infinite expansion in terms of iterated integrals of a

Using a Picard principle $y_{r,t}^{(n+1)} = 1 + \int_r^t y_{r,u}^{(n)}\,\mathrm{d}a_u$ with $y_{r,t}^{(0)} = 1$ leads to the same development.

Rem. $a_t = b \times t \Longrightarrow y_{r,t} = \sum_{k\geqslant 0} \frac{1}{k!} b^k (t-r)^k = e^{b(t-r)}$.

A remark on the importance of time

A and B are matrices

$$a_t = \int_0^t (A\mathbb{1}_{[0,1]}(s) + B\mathbb{1}_{[0,1]}(s))\,\mathrm{d}s$$

$$a'_t = \begin{cases} A & \text{if } t \in [0,1], \\ B & \text{if } t \in [1,2] \end{cases}$$

$$y'_t = y_t a'_t \text{ with } y_0 = \mathrm{Id}$$

$$\Longrightarrow y_2 = \exp(A)\exp(B)$$

$$\widehat{a}_t = \int_0^t (B\mathbb{1}_{[0,1]}(s) + A\mathbb{1}_{[0,1]}(s))\,\mathrm{d}s$$

$$\widehat{a}'_t = \begin{cases} B & \text{if } t \in [0,1], \\ A & \text{if } t \in [1,2] \end{cases}$$

$$\widehat{y}'_t = \widehat{y}_t \widehat{a}'_t \text{ with } \widehat{y}_0 = \mathrm{Id}$$

$$\Longrightarrow \widehat{y}_2 = \exp(B)\exp(A)$$

A and B are seen as directions. In general

$$\exp(A)\exp(B) = \exp(A \star_{\mathrm{BCHD}} B)$$
$$\neq \exp(B)\exp(A) = \exp(B \star_{\mathrm{BCHD}} A).$$

Alhtough $a_2 = \widehat{a}_2$, $y_2 \neq \widehat{y}_2$: integrating linear equations is really sensitive to the order of the successive directions.

Tensor product and tensor algebra

1. Set $\mathrm{U} = \mathbb{R}^d$ and $\{e_i\}_{i=1}^d$ its basis.
2. Set $\mathrm{U}^{\otimes k} = \mathrm{U} \otimes \cdots \otimes \mathrm{U}$, a tensor product of spaces.
3. Roughly speaking, $\mathrm{U}^{\otimes k}$ is $\mathrm{U} \times \cdots \times \mathrm{U}$ with a vector space structure. A basis is
$$e_J := e_{i_1} \otimes \cdots \otimes e_{i_k} \text{ with } J = i_1 \cdots i_k,\ \ i_j \in \{1, \ldots, d\}$$
4. J is a word with letters in the alphabet $\{1, \ldots, d\}$.
5. The tensor algebra is $\mathfrak{T} := \mathbb{R} \oplus \mathrm{U} \oplus \mathrm{U}^{\otimes 2} \oplus \cdots$, the superposition of the tensor spaces $\mathrm{U}^{\otimes k}$ (with $\mathrm{U}^{\otimes 0} = \mathbb{R}$), with the addition $+$ and product $\otimes$. It is naturally a graded space with a natural order for the elements. It is also a Banach algebra.
6. Set $\mathfrak{T}_k := \mathbb{R} \oplus \mathrm{U} \oplus \mathrm{U}^{\otimes 2} \oplus \cdots \mathrm{U}^{\otimes k}$, the truncation of $\mathfrak{T}$ up to order k (set the other elements of 0).
7. Alternatively, elements in $\mathfrak{T}$ are manipulated as non-commutative series/polynomials with indeterminates $e_1, \cdots, e_d$.

Chen series / signature

- Let $x \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$, $\alpha > 1/2$.
- Solve in the Banach algebra $\mathfrak{T}$ the linear equation

$$\mathbf{x}_{r,t} = 1 + \int_r^t \mathbf{x}_{r,s} \otimes \, \mathrm{d}x_s, \; t \geqslant r.$$

$\Rightarrow$ The Chen series/signature is an expansion “à la Dyson”:

$$\mathbf{x}_{r,t} = \sum_{k=0}^{+\infty} \mathbf{x}_{r,t}^{(k)} \text{ with } \mathbf{x}_{r,t}^{(0)} = 1 \text{ in } \mathbb{R}, \; \mathbf{x}_{r,t}^{(1)} = x_{r,t} \text{ in } \mathbb{R}^d$$

$$\text{and } \mathbf{x}_{r,t}^{(k)} = \sum_{J \text{ with } |J|=k} e_J \int_r^t \mathrm{d}^J x_s$$

$$\text{where } \int_r^t \mathrm{d}^{i_1 \cdots i_k} x_s := \int_r^t \int_r^{t_1} \cdots \int_r^{t_{k-1}} \mathrm{d}x_{t_k}^{i_k} \cdots \mathrm{d}x_{t_1}^{i_1}$$

Chen series: algebraic aspects

K.T. Chen '50, T. Lyons & B. Hambly 2010 / The signature $\mathbf{x}_{s,t}$ is in one-to-one (more or less) correspondence with a path $(x_r)_{r\in[s,t]}$ of finite variation.

A geometric-algebraic dictionary

Concatenation	$\mathbf{x}_{r,t} = \mathbf{x}_{r,s} \otimes \mathbf{x}_{s,t}$
Time reversal	$\mathbf{x}_{t,r}^{-1} = 1 - \int_r^t \mathrm{d}x_s \otimes \mathbf{x}_{s,r}^{-1}$ series inversion in $\mathfrak{T}$
Lie group and Lie algebra $\mathbf{x}_{r,t} = \exp(\mathbf{y}_{r,t})$	$\mathbf{y}_{r,t} = \sum_I a_I \int_r^t \mathrm{d}^I x_s$

Chen series are paths with values in $\mathfrak{T}$ (non-commutative)

$$\mathbf{x}_t := \mathbf{x}_{0,t} \text{ so that } \mathbf{x}_{s,t} = \mathbf{x}_s^{-1} \otimes \mathbf{x}_t .$$

Chen series: the second order term

Let us consider a signature $\mathbf{x}$ in $\mathfrak{T}_2$ (truncated algebra up to order 2) above a path $x \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$.

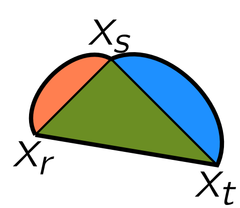
- $\mathbf{x}_t^{(1)} = x_t$: this is the path.
- $\mathbf{x}_t^{(2)}$ takes its values in $\mathbb{R}^d \otimes \mathbb{R}^d \equiv \mathrm{M}_{d\times d}(\mathbb{R})$.
- We write

$$\mathbf{s}_{s,t}^{ij} = \frac{1}{2}\left(\mathbf{x}_{s,t}^{(2),ij} + \mathbf{x}_{s,t}^{(2),ji}\right) = \frac{1}{2} x_{s,t}^i \times x_{s,t}^j, \quad \text{symmetric part}$$

$$\mathbf{a}_{s,t}^{ij} = \frac{1}{2}\left(\mathbf{x}_{s,t}^{(2),ij} - \mathbf{x}_{s,t}^{(2),ji}\right), \quad \text{anti-symmetric part}$$

- The symmetric part $\mathbf{s}_{s,t} = \frac{1}{2} x_{s,t} \otimes x_{s,t}$ is constructed from the increments $x_{s,t}$.
- The multiplication rule $\mathbf{x}_{r,t} = \mathbf{x}_{r,s} \otimes \mathbf{x}_{s,t}$ implies

$$\mathbf{a}_{r,t}^{ij} = \mathbf{a}_{r,s}^{ij} + \mathbf{a}_{s,t}^{ij} + \frac{1}{2} x_{r,s}^i \times x_{s,t}^j - \frac{1}{2} x_{r,s}^j \times x_{s,t}^i .$$

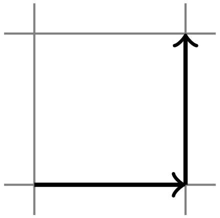


$\mathbf{a}$ stands for "Antisymmetric", also for "Area".

Chen series: an illustration

Let us consider the path x with derivative

$$\dot{x}_t = \begin{cases} e_1 & \text{if } t \in [0,1], \\ e_2 & \text{if } t \in [0,2]. \end{cases}$$



Thus $\mathbf{x}^{(1)}_{0,2} = x_{0,2} = e_1 + e_2 \equiv (1,1)$ and

$$\mathbf{x}^{(2,ij)}_{0,2} = \int_0^2 x^i_{0,s}\, \mathrm{d}x^j_{0,s}$$

$$\Longrightarrow \mathbf{x}^{(2)}_{0,2} = \frac{1}{2} e_1 \otimes e_1 + \frac{1}{2} e_2 \otimes e_2 + e_1 \otimes e_2 \equiv \begin{bmatrix} 1/2 & 0 \\ 1 & 1/2 \end{bmatrix}.$$

The anti-symmetric part is

$$\mathbf{a}^{1,2}_{0,2} = -\mathbf{a}^{2,1}_{0,2} = \frac{1}{2}.$$

This is the algebraic area of the triangle $(0,0)\,(1,0)\,(1,1)$. If we exchange the order of e_1 and e_2, $\mathbf{a}^{1,2}_{0,2}[e_2, e_1] = -1/2$.

Chen series: an illustration

Alternatively, we solve $\mathbf{x}_{0,t} = 1 + \int_0^t \mathbf{x}_{0,s} \otimes \, \mathrm{d}x_s$:

$$\text{For } t \in [0,1] \quad \mathbf{x}_{0,t} = 1 + \int_0^t \mathbf{x}_{0,s} \otimes e_1 = \exp(e_1)$$

$$\text{For } t \in [0,1] \quad \mathbf{x}_{1,t} = 1 + \int_1^t \mathbf{x}_{1,s} \otimes e_2 = \exp(e_2).$$

Using the multiplicative property,

$$\mathbf{x}_{0,2} = \mathbf{x}_{0,1} \otimes \mathbf{x}_{1,2} = \exp(e_1) \otimes \exp(e_2)$$

$$= 1 + e_1 + e_2 + \frac{1}{2} e_1 \otimes e_1 + \frac{1}{2} e_2 \otimes e_2 + e_1 \otimes e_2 + [\textbf{deg.} \geqslant 3].$$

Let a be a direction such that $\exp(a) = \mathbf{x}_{0,2}$, then

$$a = \log(\mathbf{x}_{0,2}) = x_{0,t} + \mathbf{a}^{1,2}_{0,2}[e_1, e_2] + \cdots$$

$$\text{with } [e_1, e_2] = e_1 \otimes e_2 - e_2 \otimes e_1.$$

$\Longrightarrow$ The signature encodes the Lévy area: moving in the Euclidean plane creates displacement in the other directions.

What happens for irregular paths?

For a Chen series in $\mathfrak{T}_2$ (truncated order $\leqslant 2$),

$$\mathbf{x}_{s,t} = \mathbf{x}^{(\leqslant 2)}_{s,t} := 1 + \mathbf{x}_{s,t} + \mathbf{x}^{(2)}_{s,t}, \text{ with } \mathbf{x}^{(1)}_{s,t} \in \mathbb{R}^d, \ \mathbf{x}^{(2)}_{s,t} \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

Let us assume that we are given $\mathbf{x}^{(\leqslant 2)}$ with $1/3 < \alpha \leqslant 1/2$,
$\mathbf{x}^{(1)}_{s,t} = x_{s,t} := x_t - x_s$, $|x_{s,t}| \leqslant K(t-s)^{\alpha}$ and $|\mathbf{x}^{(2)}_{s,t}| \leqslant K(t-s)^{2\alpha}$.

Could we reconstruct a whole Chen series in $\mathfrak{T}$?

Apply the multiplicative sewing lemma in $\mathfrak{T}$ (not $\mathfrak{T}_2$)

$$\begin{aligned}\mathbf{x}^{(\leqslant 2)}_{r,s,t} &:= \mathbf{x}^{(\leqslant 2)}_{r,s} \otimes \mathbf{x}^{(\leqslant 2)}_{s,t} - \mathbf{x}^{(\leqslant 2)}_{s,t} \\ &= \underbrace{\mathbf{x}^{(1)}_{r,s} \otimes \mathbf{x}^{(2)}_{s,t} + \mathbf{x}^{(2)}_{r,s} \otimes \mathbf{x}^{(2)}_{s,t} + \mathbf{x}^{(2)}_{r,s} \otimes \mathbf{x}^{(1)}_{s,t}}_{\text{only terms of order } \geqslant 3}.\end{aligned}$$

$$\implies |\mathbf{x}^{(\leqslant 2)}_{r,s,t}| \leqslant 3K(t-s)^{3\alpha} \ ☺$$

The Lyons expansion theorem

T. Lyons 1998/ Knowing a path $\mathbf{x}^{(\leqslant k)}$ with values in $\mathfrak{T}_k = \mathbb{R} \oplus \mathrm{U} \oplus \cdots \oplus \mathrm{U}^{\otimes k}$ satisfying

$$\mathbf{x}_{s,t}^{(\leqslant k,1)} = x_{s,t} := x_t - x_s \text{ with } x \in \mathcal{C}^\alpha([0,T],\mathrm{U}),$$

$$|\mathbf{x}_{s,t}^{(\leqslant k,i)}| \leqslant K(t-s)^{i\alpha},\ i \leqslant k,\ \frac{1}{\alpha} \leqslant k+1$$

is sufficient to construct a path $\mathbf{x}$ with values in $\mathfrak{T}$ with $\mathbf{x}^{(\leqslant k,i)} = \mathbf{x}^{(i)}$, $i \leqslant k$.

Besides, $\mathbf{x}^{(\leqslant k)} \mapsto \mathbf{x}$ is continuous.

For $1/3 < \alpha \leqslant 1/2$, we recover that the area **a** is a necessary but hidden information. Knowing the path x itself is not sufficient unless $\alpha > 1/2$.

What is a rough path?

Def: A rough path of order k above a path $x \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$ is a path $\mathbf{x}$ with values in $\mathfrak{T}_k$ satisfying

$$\mathbf{x}^{(1)}_{s,t} = x_{s,t}$$

$$|\mathbf{x}^{(i)}_{s,t}| \leqslant K(t-s)^{i\alpha},\ i \leqslant k,\ \frac{1}{\alpha} \leqslant k+1$$

1. The are an infinite number of rough path above a path x:
$$\mathbf{y}_{s,t} = \mathbf{x}_{s,t} + \phi_t - \phi_s,\ \forall \phi \in \mathcal{C}^{2\alpha}([0,T],\mathbb{R}^d \otimes \mathbb{R}^d)$$
is also a rough path above x ($1/3 < \alpha \leqslant 1/2$).
2. When $x \in \mathcal{C}^\alpha$, $1/2 < \alpha$, $\Lambda(x)_t = 1 + x_t + \int_0^t x_{0,r} \otimes \,\mathrm{d}x_r$ is a rough path in $\mathfrak{T}_2$.
3. Rough paths $\mathbf{x}$ above $x \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$ for which there exists $x^n \in \mathcal{C}^\infty([0,T],\mathbb{R}^d)$ with $\Lambda(x^n)$ converges to $\mathbf{x}$ in $\mathcal{C}^\beta$, $\beta < \alpha$, are called geometric rough paths.
4. Not all rough paths are geometric ones (Itô vs Stratonovich).

To summarize

1. A rough path is a path with values in a non-commutative, tensor space.

2. A rough path is a natural extension of truncated Chen series/signatures to irregular paths.

3. There exist an infinite number of rough paths above a given path.

4. There is no canonical way to construct a rough path above a given path, even if there are natural ones (it is however always possible, a non-trivial result: T. Lyons & N. Victoir 2007, J. Unterberger 2010, M. Hairer 2013).

5. The signature translates geometric properties to algebraic ones.

— VI —

Rough Differential Equations:
The Davie’s approach

Differential equations

- $x \in \mathcal{C}^1([0,T],\mathbb{R}^d)$
- $\mathbf{x} = \Phi(x)$, natural lift of x
- $f = (f_1, \ldots, f_d)$ smooth functions from $\mathbb{R}^d$ to $\mathbb{R}^m$.

Differential equation driven by a path

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s = a + \sum_{i=1}^d \int_0^t f_i(y_s)\,\mathrm{d}x_s^i,$$

$$a, y_t \in \mathbb{R}^m.$$

This ODE may be thought as obtained by a “combination” of d flows with coefficients given by the speed $\dot{x}_t^i$.

Differential Equations

Newton formula $\Longrightarrow$ for $g \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$,

$$g^k(y_t) = g^k(a) + \sum_{\substack{i=1,\dots,d \\ j=1,\dots,m}} \int_0^t \partial_j f_i^j g^j(y_s)\,\mathrm{d}x_s^i, \quad k=1,\dots,m,$$

$$g(y_t) = g(a) + \int_0^t f g(y_s)\,\mathrm{d}x_s$$

by identifying f with a vector field: for $g \in \mathcal{C}^1,\ b \in \mathbb{R}^d$,

$$f g(a) \cdot b := \sum_{\substack{j=1,\dots,m \\ i=1,\dots,d}} f_i^j(a)\partial_j g(a) b^i.$$

Thus, with $E =$identity,

$$y_t = a + f E(a) x_{0,t} + \int_0^t (f E(y_s) - f E(a))\,\mathrm{d}x_s$$
$$= a + f E(a)\mathbf{x}_{0,t}^{(1)} + f f E(a)\mathbf{x}_{0,t}^{(2)} + f f f E(a)\mathbf{x}_{0,t}^{(3)} + \cdots$$

$\Longrightarrow$ The solution y_t may be computed from the signature.

Rough differential equations: Davie's approach

The development suggests to set (with $E(a) = a$)

$$\phi_{t,s}(a) = f^{[0]}(a) + f^{[1]}\mathbf{x}^{(1)}_{s,t} + ffE(a)\mathbf{x}^{(2)}_{s,t}$$

$$\psi_{t,s}(a) = a + \phi_{t,s}(a)$$

$$\text{with } f^{[0]} = E,\ f^{[1]} = fE \text{ and } f^{[2]} = ffE$$

as an approximation of y_t when $y_s = a$ and $f \in \mathcal{C}^{1+\gamma}$ with bounded derivatives, $\alpha(2+\gamma) > 1$.

Using $\mathbf{x}^{(2)}_{r,t} = \mathbf{x}^{(2)}_{r,s} + \mathbf{x}^{(2)}_{s,t} + \mathbf{x}^{(1)}_{r,s} \otimes \mathbf{x}^{(1)}_{s,t}$,

$$\begin{aligned}\psi_{t,s} \circ \psi_{s,r}(a) = {} & \overbrace{a + f^{[1]}(a)\mathbf{x}^{(1)}_{r,t} + f^{[2]}(a)\mathbf{x}^{(2)}_{r,t}}^{\psi_{t,r}(a)} \\ & + (f^{[2]}(\psi_{s,r}(a)) - f^{[2]}(a))\mathbf{x}^{(2)}_{s,t} \\ & + (f^{[1]}(\psi_{s,r}(a)) - f^{[1]}(a))\mathbf{x}^{(1)}_{s,t} - f^{[2]}(a)\mathbf{x}^{(1)}_{r,s} \otimes \mathbf{x}^{(1)}_{s,t}.\end{aligned}$$

Dealing with the green and red terms

Thus

$$|(f^{[2]}(\psi_{s,r}(a)) - f^{[2]}(a))\mathbf{x}^{(2)}_{s,t}| \leqslant C|t-r|^{(2+\gamma)\alpha}. \quad ☺$$

Since $\mathbf{x}^{(1)}_{s,t} = x_{s,t}$,

$$\begin{aligned}
&|(f^{[1]}(\phi_{s,r}(a)) - f^{[1]}(a))\mathbf{x}^{(1)}_{s,t} - f^{[2]}(a)\mathbf{x}^{(1)}_{r,s} \otimes \mathbf{x}^{(1)}_{s,t}| \\
&\leqslant |(f^{[1]}(a + f^{[1]}(a)x_{r,s}) - f^{[1]}(a))x_{s,t} - f^{[2]}(a)x_{r,s} \otimes x_{s,t}| \\
&\qquad + \underbrace{|(f^{[1]}(a + f^{[1]}(a)x_{r,s}) - f^{[1]}(\psi_{r,s}(a)))x_{s,t}|}_{\leqslant C|t-r|^{3\alpha}\ ☺}
\end{aligned}$$

It remains to control the violet term.

Dealing with the violet term

With a Taylor development,

$$
\begin{aligned}
&|((f^{[1]}(a + f^{[1]}(a)x_{r,s}) - f^{[1]}(a))x_{s,t} - f^{[2]}(a)x_{r,s} \otimes x_{s,t}| \\
&= \left| \int_0^1 f^{[1]}(a + \tau f^{[1]}(a)x_{r,s}) f^{[1]}(a)x_{r,s} \otimes x_{s,t} \, \mathrm{d}\tau - f^{[2]}(a)x_{r,s} \otimes x_{s,t} \right| \\
&= \left| \int_0^1 (f^{[1]}(a + \tau f^{[1]}(a)x_{r,s}) - f^{[1]}(a)) f^{[1]}(a)x_{r,s} \otimes x_{s,t} \, \mathrm{d}\tau \right| \\
&\leqslant K|t - r|^{(2+\gamma)\alpha} ☺
\end{aligned}
$$

since $|x_{s,t}| \leqslant C|t - s|^{\alpha}$.

We have then proved that

$$|\psi_{t,s,r}(a)| := |\psi_{t,s} \circ \psi_{s,r}(a) - \psi_{t,r}(a)| \leqslant K|t - r|^{(2+\gamma)\alpha} ☺$$

RDE through the Davie approach

For a partition t_i, we set the Euler scheme

$$y_{i+1} := \psi_{i+1,i}(y_i) \text{ where indice } i \text{ means } t_i.$$

For $i < j < k$,

$$\begin{aligned} & y_k - \psi_{k,i}(y_i) \\ & \quad = y_k - \psi_{k,j}(y_j) + \psi_{k,j}(y_j) - \psi_{k,j}(\psi_{j,i}(y)) + \psi_{k,j,i}(y_i). \end{aligned}$$

Thus,

$$V_{k,i} := |y_k - \psi_{k,i}(y_i)| \leqslant V_{k,j} + \|\psi_{k,j}\|_{\mathrm{Lip}} V_{k,j} + K|t_k - t_i|^{(2+\gamma)\alpha}$$

A variant of the discrete additive sewing lemma shows that

$$|y_k - \psi_{k,i}(y_i)| \leqslant L|t_k - t_i|^{(2+\gamma)\alpha}, \; L \text{ is uniform in } \pi$$

RDE through the Davie approach

The solution of the Euler scheme $y_{i+1} = \psi_{i+1,i}(y_i)$ satisfies

$$|y_k - \psi_{k,i}(y_i)| \leqslant L|t_k - t_i|^{(2+\gamma)\alpha}$$

where L does not depend on the partition.

With the Ascoli-Arzelà theorem, there exists $y \in \mathcal{C}([0,T],\mathbb{R}^d)$ s.t.

$$|y_t - y_s - f^{[1]}(y_s)x_{s,t} - f^{[2]}(y_s)\mathbf{x}^{(2)}_{s,t}| \leqslant L|t-s|^{(2+\gamma)\alpha}. \qquad (\clubsuit)$$

A.M. Davie 2010/ If $f \in \mathcal{C}^{1+\gamma}$, $\alpha(2+\gamma) > 1$, then there exists a path $y \in \mathcal{C}^{\alpha}([0,T],\mathbb{R}^d)$ that solves (♣).

If $f \in \mathcal{C}^{2+\gamma}$, $\alpha(2+\gamma) > 1$, then y is unique and locally Lipschitz continuous wrt initial point.

The Non-Linear Sewing Lemma

A general framework allows one to deal with flows, without reference to a particular approximation (I. Bailleul 2014).

A. Brault & AL, 2017/ Let $\psi_{t,s}$ be a family satisfying with $\delta_T \to 0$ as $T \to 0$,

$$\|\psi_{t,s} - E\|_\infty \leqslant \delta_{t-s},\ \|\psi_{t,s}\|_{\mathrm{Lip}} \leqslant 1 + \delta_{t-s}$$
$$\|\psi_{t,s} \circ \psi_{s,r} - \psi_{t,r}\|_\infty \leqslant C|t-r|^\theta,\ \theta > 1.$$

Then there exists a measurable flow χ satisfying

$$\chi_{t,s} \circ \chi_{s,r} = \chi_{t,r} \text{ and } \|\chi_{t,s} - \psi_{t,s}\|_\infty \leqslant C|t-s|^\theta.$$

I. Bailleul, A. Brault & AL/ Using some more stringent conditions, the flow is unique and Lipschitz.

On Davie's solution

1. When $f \in \mathcal{C}^{1+\gamma}$, the solution is not necessarily unique (A.M. Davie 2010). Counter-examples exist.
2. When $f \in \mathcal{C}^{2+\gamma}$, the rate of convergence of the Euler scheme could be given.
3. The dichotomy the regularity $1+\gamma$ (Cauchy-Peaono) and $2+\gamma$ (Cauchy-Lipschitz) is "essentially" sharp.
4. There are other ways to approximate flows leading to higher order rate of convergence
   - I. Bailleul approximation by solving ODE by computing formal logarithm
   - P. Friz & N. Victoir using sub-Riemannian geodesics
5. The solution is defined as a path $y \in \mathcal{C}^\alpha([0,T],\mathbb{R}^m)$. This solution cannot be used as an input for another integral. This draws the line between $\alpha > 1/2$ and $\alpha \leqslant 1/2$.

— VII —

Rough Differential Equations: Gubinelli’s derivative

The idea of Gubinelli's derivative

When y is a solution in the sense of Davie of $y_t = a + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s$,

$$|y_{s,t} - f^{[1]}(y_s)x_{s,t} - f^{[2]}(y_s)\mathbf{x}^{(2)}_{s,t}| \leqslant L|t-s|^{(2+\gamma)\alpha}$$

with $f^{[1]} = fE$ and $f^{[2]} = ffE = f\nabla f$.

We rewrite

$$y_{s,t} = y^{\dagger}_s x_{s,t} + y^{\sharp}_{s,t} \text{ with } \begin{cases} y^{\dagger}_s = f^{[1]}(y_s) \\ y^{\sharp}_{s,t} = f^{[2]}(y_s)\mathbf{x}^{(2)}_{s,t} + \text{remain.} \end{cases}$$

so that $|y^{\sharp}_{s,t}| \leqslant C|t-s|^{2\alpha}$.

Similarly to Young integrals, increments of the solutions locally "look like" the one of x.

Gubinelli's space

Def. For $1/3 < \alpha \leqslant 1/2$, $0 < \gamma \leqslant 1$ and a path $x \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$, the Gubinelli space is

$$\mathbb{G}_x = \left\{ (y, y^\dagger) \;\middle|\; \begin{array}{l} y \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d) \\ y^\dagger \in \mathcal{C}^{\alpha\gamma}([0,T],\mathrm{L}(\mathbb{R}^d,\mathbb{R}^m)) \\ y_{s,t} = y^\dagger_s \cdot x_{s,t} + y^\sharp_{s,t} \text{ with } |y^\sharp_{s,t}| \leqslant C|t-s|^{2\alpha} \end{array} \right\}.$$

The idea is to encode both the paths and its increment wrt x:

- $y^\dagger$ is the Gubinelli derivative
- $y^\sharp$ is the remainder

The space $\mathbb{G}_x$ has a vector space structure. It depends on x (this has heavy consequences).

Def. An element of $\mathbb{G}_x$ is called a controlled rough path.

Gubinelli's space and integration

The notion of Gubinelli's space is convenient for integration.

M. Gubinelli 2004/ Assume $\theta = \alpha(2+\gamma) > 1$. Let $\mathbf{x}$ be a rough path above x. Let $\underline{y} = (y, y^\dagger) \in \mathbb{G}_x$. Then

$$\int_r^t \underline{y}_s \, \mathrm{d}\mathbf{x}_s = \lim_{|\pi|\to 0} \sum_{\pi\cap[r,t]} \left(y_u x_{u,v} + y_u^\dagger \mathbf{x}_{u,v}^{(2)} \right)$$

is well defined and linear with respect to $(y, y^\dagger)$. Besides,

$$\left| \int_r^t \underline{y}_s \, \mathrm{d}\mathbf{x}_s - y_r x_{r,t} - y_r^\dagger \mathbf{x}_{r,t}^{(2)} \right| \leqslant K|t-r|^\theta.$$

This linear rough integral, which is a natural expansion of the Young integral (if $\alpha > 1/2$, $\sum y_u^\dagger \mathbf{x}_{u,v}^{(2)}$ disappear as $|\pi| \to 0$).

Gubinelli's space and integration

Again, the existence of the rough integral is an immediate consequence of ... (surprise) ... the Additive Sewing Lemma. Set

$$Z_{s,t} = y_s x_{s,t} + y_s^{\dagger} \mathbf{x}^{(2)}_{s,t}.$$

Thus

$$\begin{aligned} Z_{r,s} + Z_{s,t} - Z_{r,t} &= -y_r^{\dagger} x_{r,s} \otimes x_{s,t} + y_{r,s} x_{s,t} + y^{\dagger}_{r,s} \mathbf{x}^{(2)}_{s,t} \\ &= y^{\sharp}_{r,s} x_{s,t} + y^{\dagger}_{r,s} \mathbf{x}^{(2)}_{s,t} \end{aligned}$$

Acutally,

$$|y^{\dagger}_{r,s} \mathbf{x}^{(2)}_{s,t}| \leqslant C|t-r|^{\alpha(2+\gamma)} \; ☺$$

$$|y^{\sharp}_{r,s} x_{s,t}| \leqslant C|t-r|^{3\alpha} \; ☺.$$

The Omega lemma for Gubinelli's space

Let $\underline{y} = (y, y^\dagger) \in \mathbb{G}_x$ and $f \in \mathcal{C}_{\mathrm{b}}^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^m)$.

Using the Taylor development,

$$f(y_t) - f(y_s) = \overbrace{\nabla f(y_s) y_s^\dagger}^{f(y)_t^\dagger} x_{s,t} + \underbrace{\nabla f(y_s) y_{s,t}^\sharp + \int_0^1 (\nabla f(y_s + \tau y_{s,t}) - \nabla f(y_s)) y_{s,t} \,\mathrm{d}\tau}_{f(y)_{s,t}^\sharp}$$

The Omega lemma for Gubinelli's space

Omega lemma, L. Coutin & AL 2018/

When $f \in \mathcal{C}_{\mathrm{b}}^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^m)$, $\alpha(1+\gamma) > 1$, then

$$\Omega f(\underline{y}) = (f(y_t), \nabla f(y_t) y_t^{\dagger})_{t \in [0,T]}$$

maps $\mathbb{G}_x$ to $\mathbb{G}_x$.

If $f \in \mathcal{C}_{\mathrm{b}}^{1+k+\gamma}$, $k \geqslant 0$, then Ωf is locally in $\mathcal{C}^{k+(1-\kappa)\gamma}$ up to decreasing the regularity of the arrival space.

Consequences on RDE

We apply the same techniques as for YDE when $\alpha(2+\gamma) > 1$. However, f has "one rank" of regularity more than for YDE

1. Existence (but not uniqueness) for $f \in \mathcal{C}^{1+\gamma}$.
2. Uniqueness for $f \in \mathcal{C}^{2+\gamma}$ and local Hölder regularity of the solutions wrt (a, f).
3. Convergence of the Euler scheme for $f \in \mathcal{C}^{2+\gamma}$.
4. The Itô map $\mathbf{x} \mapsto y$ is locally Lipschitz continuous.
5. Sensitivity wrt $\mathbf{x}$ is a cumbersome issue. The "direction" should be defined.
6. When $\mathbf{x}$ is a geometric rough path, y is the limit of solution of ODEs constructed by approximations of x.
7. Global existence ($T = +\infty$) is also cumbersome and depends on the notion of solution.
8. ...

To summarize (Rough case, $1/3 < \alpha \leqslant 1/2$)

	ODE	RDE
"Cauchy-Peano" existence non uniqueness *a priori*	f continuous	$f \in \mathcal{C}^{1+\gamma}$ $\alpha(2+\gamma) > 1$ sol in $\mathbb{G}_x$
"Cauchy-Lipschitz" uniqueness Lipschitz flow conv. of Euler scheme	f Lipschitz $f \in \mathcal{C}^k \Rightarrow \mathcal{I} \in \mathcal{C}^k$	$f \in \mathcal{C}^{2+\gamma}$ $\alpha(2+\gamma) > 1$ $\Rightarrow \mathcal{I} : \mathbb{G}_x \to \mathbb{G}_x$ $\mathcal{I}$ is loc. Lipschitz wrt $\mathbf{x}$ $f \in \mathcal{C}^{k+1+\gamma} \Rightarrow \mathcal{I} \in \mathcal{C}^{k+\epsilon\gamma}$ for $k \geqslant 1$ wrt (a, f).

- This classification is essentially sharp.
- Several solutions exist unless $f \in \mathcal{C}^{2+\gamma}$ (A.M. Davie 2010).
- Any solution can be lifted as a rough path/an element of $\mathbb{G}_x$.

Other approaches

- The original approach from T. Lyons consists in finding the fixed point as a rough path lying above (y, x), hence constructing the iterated integrals between y and x.

  There is no need to compute all the iterated integrals, one may consider constructing only y and $\int y \, \mathrm{d}x$ (AL & N. Victoir 2006).

- There are several ways to construct flows
  - A.M. Davie by Euler-type approximation
  - I. Bailleul by “freezing” the vector field
  - P. Friz & N. Victor by using sub-Riemannian geodesics

- Approaches for SPDE work as well
  - M. Hairer using regularity structures.
  - M. Gubinelli, N. Perkowski & P. Imkeller using paraproducts

Application to stochastic analysis

$$\mathbf{B}^{(2)}_{r,t} = \int_r^t (B_s - B_r) \otimes \circ \, \mathrm{d}B_s \implies \text{RDE} \equiv \text{Stratonovich SDE}$$

$$\mathbf{B}^{(2)}_{r,t} = \int_r^t (B_s - B_r) \otimes \, \mathrm{d}B_s \implies \text{RDE} \equiv \text{Itô SDE}.$$

Applied to SDE, we may

- Prove large deviation using the transfer principle by focusing on the enhanced Brownian motion
- Prove support theorem
- Justify some Monte Carlo simulation
- Understand some limit theorems (Wong-Zakai, McShane, homogenization, ...)

Applied to fractional Brownian motion, many results known from SDE could be extended (existence of density, ...).

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